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Distribution of the Number of Poisson Points in Poisson Voronoi Tessellation

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Abstract

Let two independent sets Φ_1 and Φ_2 follow homogeneous PPP having different densities in two dimensional space. Assuming that Voronoi Tessellations are generated with respect to Φ_1. This technical report is an analytical calculation of the probability distribution of Φ_2 cardinality at an arbitrary tessellation of Φ_1.

1 Introduction

Voronoi Tessellation is a widely used mathematical framework for space subdivision in random partitions. It is used in a variety of fields such us statistical mechanics, quantum field theory, astrophysics, telecommunications, social networks, biology etc. On the other hand, Poisson Point Process (PPP) is one of the most common tools at stochastic geometry, since it provides suitable mathematical models and appropriate statistical methods to analyze macroscopic properties by averaging all possible microstates. In this work we study and solve the following problem: Let two independent sets Φ_1 and Φ_2 follow homogeneous PPP having different densities in two dimensional space assuming that Voronoi Tessellations are generated with respect to Φ_1. What is the probability distribution of Φ_2 cardinality at an arbitrary tessellation of Φ_1? Furthermore, we calculate the first and the second moments of the aforementioned probability distribution. In our problem, Φ_1 represents the positions of telecommunication Base Stations (BSs) and Φ_2 the position of the Users.

2 Base Stations and Users topology

We model positions of the BSs and Users as a homogeneous Poisson Point Process. So, if the density of users (or BSs) at a certain area A is λ then the Number N of them is a random variable and it given from

$$P(N = k \mid A) = \frac{(\lambda A)^k e^{-\lambda A}}{k!}, \quad k = 0, 1, \ldots \tag{1}$$

Homogeneous, means that after the chosen of the number of BSs at certain aria A, their locations follows uniform distribution at $2D$ space.

3 Distribution of the Cell Size

A given set of centers can divide the space to specific regions, known as Voronoi Tessellations (or Voronoi Regions). Each of them contains those points of space that are closest to the same center. In our case, the centers is the location of the BSs so Voronoi tessellation represents the aria of coverage of each one of them (we assume that all the BS have the same transmit power). At the particular case, where the centers are randomly and uncorrelated distributed, is called Poisson Voronoi Tessellation (PVT), see Figure 1.

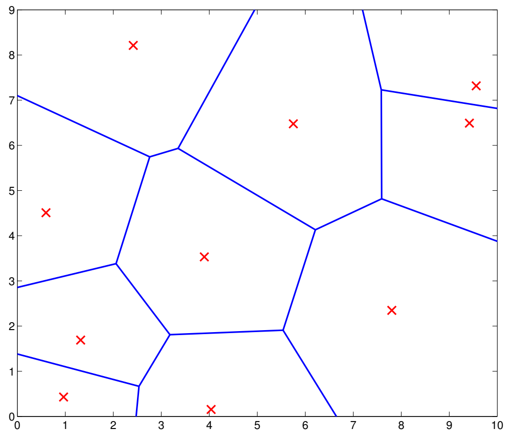

Figure 1: Voronoi Tessellation example, with 10 centers at an aria of $100m^2$

We are interested about the PDF of Voronoi cells size (2D), if the number BSs follows homogeneous PPP. Unfortunately this is an open mathematical problem and does not exist any close form solution until today. However there exist several PDFs that provide an approximate numerical solution, based of the Gamma distribution.

$$g(x; a, b, c) = \frac{ab^{\frac{c}{a}} x^{c-1} e^{-bx^a}}{\Gamma(\frac{c}{a})} \tag{2}$$

Note that $x = \frac{S}{\langle S \rangle}$, S is the specific size of a certain cell and $\langle S \rangle$ is the average size of the all Voronoi Regions. For more compact notation for our problem we can write $x = S\lambda_{BS}$, where now λ_{BS} is the density of the BS. a, b, c are fitting parameters. At [1], we take the results of the fitting parameters $a = 1.07950,\ b = 3.03226,\ c = 3.31122$. In [2] the same author present us the best fitting results if we fix $a = 1$, which is $b = 3.52418$ and $c = 3.52440$. We calculate the difference between them $\delta^{3,2} = |g(x; a, b, c) - g(x; a = 1, b, c)|$, the average absolute difference is $\delta_{av}^{3,2} = 9.8977e-04$ and the maximum difference is $\delta_{max}^{3,2} = 0.0103$.To have an intuition about the result, means that the maximum difference if roughly 1% of the $\langle S \rangle$. See Figure 2. The superscript 3,2 denotes the comparison between models with 3 and 2 and fitting parameters respectively.

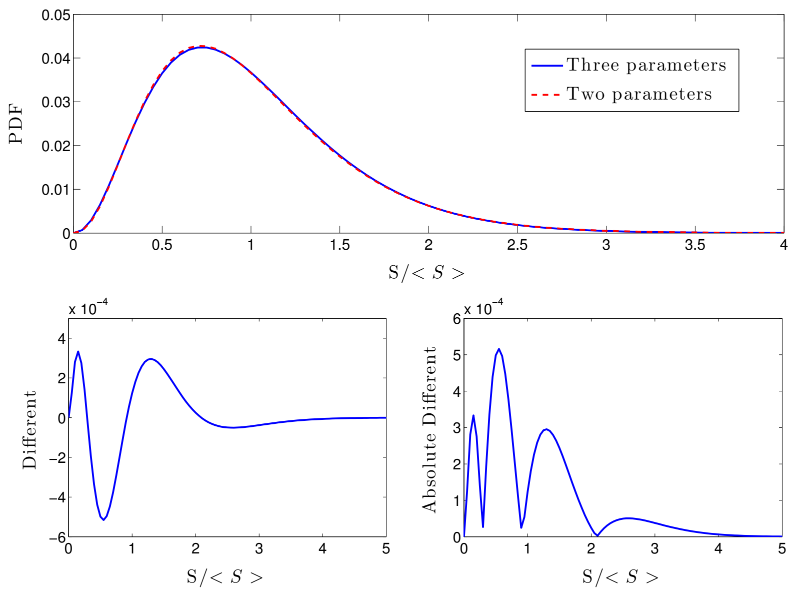

Figure 2: difference between three and two parameters fitting models

At [3], a simpler distribution has been proposed taking into account only the dimensions of the space.

$$f(x; d) = \frac{[\frac{3d+1}{2}]^{(3d+1)/2} x^{\frac{3d-1}{2}} e^{\frac{-(3d+1)x}{2}}}{\Gamma(\frac{3d+1}{2})} \tag{3}$$

Where d is number of dimensions of the space. In our case $d = 2$ and again $x = S\lambda_{BS}$. So the result for our case is the same if you set at equation 2, $a = 1,\ b = c = 3.5$. So we get

$$f(S) = \frac{343}{15}\sqrt{\frac{7}{2\pi}}(\lambda_{BS}S)^{\frac{5}{2}} e^{-\frac{7}{2}\lambda_{BS}S}\lambda_{BS} \tag{4}$$

Which is simpler than above. It is obvious that there is a trade off between accuracy and complexity. We compute the difference $\delta^{3,1} = |g(x; a, b, c) - f(x; d =$

2)|. The average difference is $\delta_{av}^{3,1} = 0.0011$ and the maximum difference is $\delta_{max}^{3,1} = 0.0108$, Figure 3. Which are almost the same as the case of tow parameters. For higher dimension spaces this accuracy does not hold.

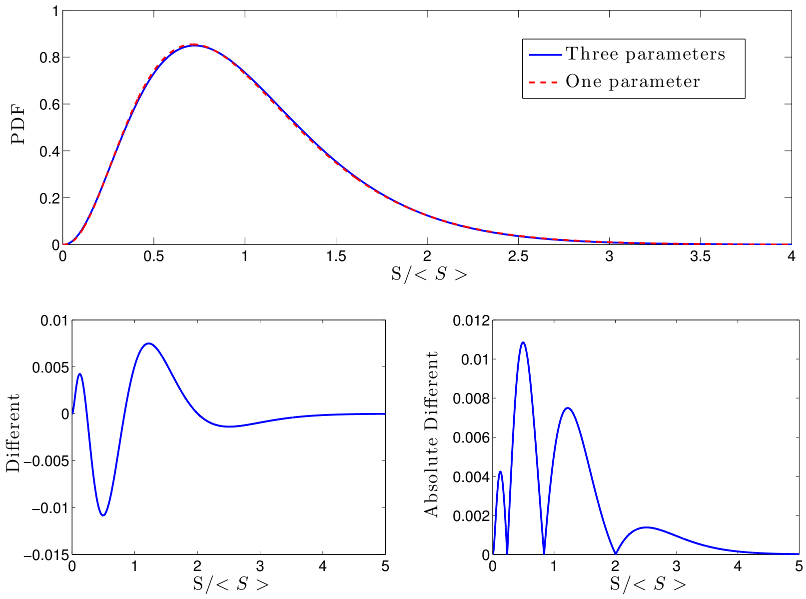

Figure 3: difference between three and one parameters fitting models

4 PDF of Number of users in a Cell

As the users are distributed as a homogeneous PPP (the following results holds and for the non-homogeneous case), as we see above

$$P(N_u = k \mid S) = \frac{(\lambda_u S)^k e^{-\lambda_u S}}{k!}, \quad k = 0, 1, \ldots \tag{5}$$

Thur $P(N_u = k)$ is calculating from

$$P(N_u = k) = \int_0^\infty P(N_u = k \mid S) f(S) dS \tag{6}$$

$$P(N_u = k) = \int_0^\infty \frac{(\lambda_u S)^k e^{-\lambda_u S}}{k!} \frac{343}{15} \sqrt{\frac{7}{2\pi}} (\lambda_{BS} S)^{\frac{5}{2}} e^{-\frac{7}{2}\lambda_{BS} S} \lambda_{BS} dS \tag{7}$$

$$P(N_u = k) = \frac{\lambda_u^k}{k!} \frac{343}{15} \sqrt{\frac{7}{2\pi}} \lambda_{BS}^{\frac{7}{2}} \int_0^\infty S^{k+\frac{5}{2}} e^{-(\lambda_u + \frac{7}{2}\lambda_{BS})S} dS \tag{8}$$

We set $\mathcal{A}(k) = \frac{\lambda_u^k}{k!} \frac{343}{15} \sqrt{\frac{7}{2\pi}} \lambda_{BS}^{\frac{7}{2}}$.

$$P(N_u = k) = \mathcal{A}(k) \int_0^\infty S^{k+\frac{5}{2}} e^{-(\lambda_u + \frac{7}{2}\lambda_{BS})S} dS \tag{9}$$

The calculation of the integral is not so trivial, so we will present the basic steps. Which give us a better understanding of the solution. First of all we set $\beta = \lambda_u + \frac{7}{2}\lambda_{BS}$ After $k+2$ integrations by parts we get

$$\int_0^\infty S^{k+\frac{5}{2}} e^{-\beta S} dS = \frac{\left(\frac{1}{\beta}\right)^{k+2} \Gamma(k + \frac{7}{2})}{\sqrt{\pi}} \int_0^\infty S^{\frac{1}{2}} e^{-\beta S} dS\ . \tag{10}$$

We continue with the calculation of the new integral by set a new variable $u^2 = S$ and $dS = 2udu$

$$\int_0^\infty u e^{-\beta u^2} 2udu = -2 \int_0^\infty -u^2 e^{-\beta u^2} du\ , \tag{11}$$

note that we can write the integral argument as derivative of constant β, $-u^2 e^{-\beta u^2} = \frac{\partial}{\partial \beta} e^{-\beta u^2}$ so we get

$$\int_0^\infty u e^{-\beta u^2} 2udu = -2 \int_0^\infty \frac{\partial}{\partial \beta} e^{-\beta u^2} du \tag{12}$$

$$= -2 \frac{\partial}{\partial \beta} \int_0^\infty e^{-\beta u^2} du\ , \tag{13}$$

where is a Gaussian integral so

$$\int_0^\infty S^{\frac{1}{2}} e^{-\beta S} dS = -2 \frac{\partial}{\partial \beta} \frac{\sqrt{\pi}}{2\sqrt{\beta}} \tag{14}$$

$$= \frac{1}{2}\sqrt{\pi}\beta^{-3/2} \; , \tag{15}$$

finally, if we combine them

$$P(N = k) = \mathcal{A}(k) \left(\frac{1}{\beta}\right)^{k+\frac{7}{2}} \Gamma(k + \frac{7}{2}) \; . \tag{16}$$

or

$$P(N_u = k) = \frac{343}{k!15} \sqrt{\frac{7}{2\pi}} \frac{\lambda_{BS}^{\frac{7}{2}} \lambda_u^k}{(\lambda_u + \frac{7}{2}\lambda_{BS})^{k+\frac{7}{2}}} \Gamma(k + \frac{7}{2}) \tag{17}$$

If we derive both numerator and denominator by $\lambda_{BS}^{k+\frac{7}{2}}$, we take an expression which depends only at the ratio between of the users and BS density $\rho = \frac{\lambda_u}{\lambda_{BS}}$, which is nice!

$$P(N_u = k) = \frac{343}{k!15} \sqrt{\frac{7}{2\pi}} \frac{\rho^k}{(\rho + \frac{7}{2})^{k+\frac{7}{2}}} \Gamma(k + \frac{7}{2}) \tag{18}$$

From "reference of equation" and after some calculations, the probability of having "zero" points at one PVR is $P(N_u = 0) = P_0 = \frac{343}{8\alpha^{7/2}}\sqrt{\frac{7}{2}}$. Thus we re-write the last result in a more intuitive way. At the Figures 4 and 5 we see PDF and CDF for different values of ratio $\rho = \frac{\lambda_u}{\lambda_{BS}}$.

$$P(N_u = k) = \left(\frac{\rho^k}{k!(\rho + 7/2)^k} \prod_{n=1}^{k} (n + \frac{5}{2}) \right) P_0 \tag{19}$$

4.1 First and Second Moments of the Distribution

We calculate the Mean and the Variance of the Number of Poisson Points in Poisson Voronoi Tessellation. By equation 17 we calculate the average number of points and their variance in a random PVT by $\langle k \rangle = \sum_{k=0}^{\inf} k \cdot P(N_u = k)$ and $Var_k = \langle k^2 \rangle - \langle k \rangle^2$. Hopefully the series converge, so first and second moments

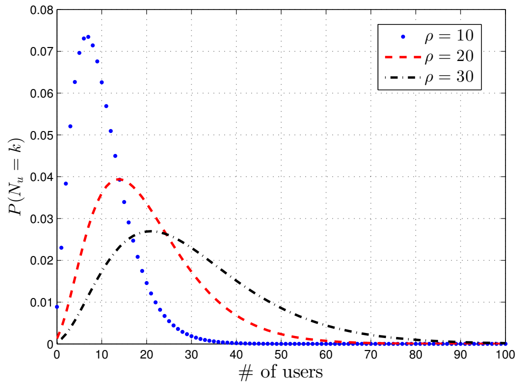

Figure 4: PDF for different values of ratio $\rho = \frac{\lambda_u}{\lambda_{BS}}$

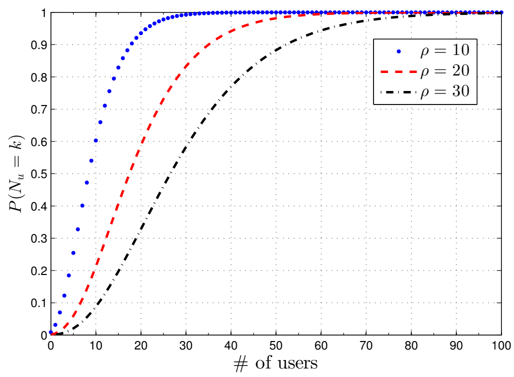

Figure 5: CDF for different values of ratio $\rho = \frac{\lambda_u}{\lambda_{BS}}$

of the distribution are

$$\langle k \rangle = \rho \qquad \text{and} \qquad \mathrm{var}_k = \rho + \frac{2}{7}\rho^2 \; . \tag{20}$$

From equation 20, we observe that the variance of the number of users within a cell drops quadratically w.r.t the density of deployed BSs, but the mean drops to. The coefficient variation is greater than 1. So the relative variance of points in a voronoi cell is not decreasing by the rising of BS density.

5 Approximation of the result

If we take into account the asymptotic of gamma function $\lim_{n\to 0}\frac{\Gamma(n+\alpha)}{\Gamma(n)n^{\alpha}} = 1$ and the definition $\Gamma(k) = (k-1)!$, eq. 18 is significant simplified to

$$P(N = k) = A_{\rho} u_{\rho}^{k} k^{5/2} \; , \tag{21}$$

where $u_{\rho} = \frac{\rho}{\rho+7/2}$. Due to the asymptotic approach of gamma function the pdf lost its normalization. So, A_{ρ} is the new normalization factor which depends on ρ. Fig. 6 provides an analytical fit to the normalization factor and Fig. 7 shows the square error between the Approximation and the distribution of users cardinality for $\rho = 30$.

The simpler analytical form eq. 21 allows not only further theoretical use of the result but also provides much wider computing operability range instead of eq. 18. e.x. eq. 18 in the process to calculate probability having $k = 169$ users in a cell (independent of ρ) exceeds $1.7977e+308$ digits (largest finite floating-point number in IEEE double precision), at the same time eq. 21 for the worst case $\rho = 1$ does not exceed $4.0466e + 111$ and for more reasonable values e.x. $\rho = 20$ the needed floating-points is $7.9601e + 09$.

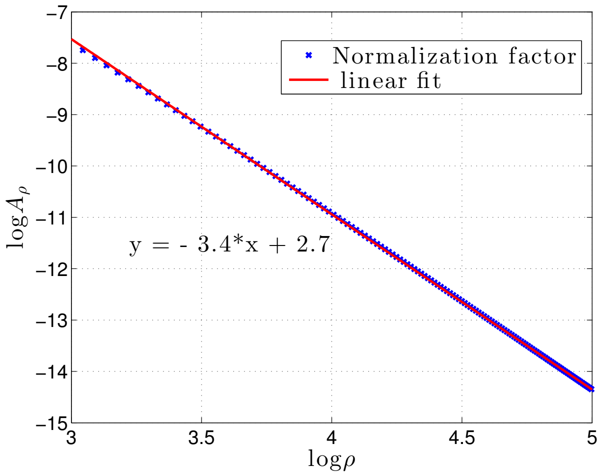

Figure 6: Fit to normalization factor w.r.t ρ

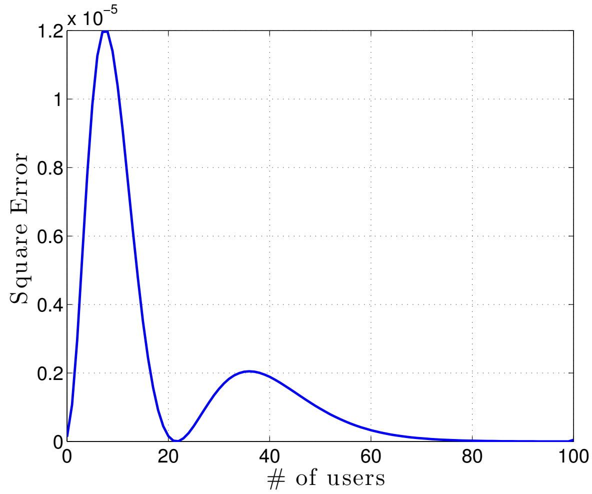

Figure 7: Square Error of Approximation for $\rho = 30$

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